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A Note on the Degree of Approximation with an Optimal, Discrete Polynomial

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A saturation theorem and an asymptotic theorem are proved for an optimal, discrete, positive algebraic polynomial operator. The operator is based on the Gauss–Legendre quadrature formula.

1. INTRODUCTION

Let $\{P_k\}$ be the sequence of Legendre polynomials, orthogonal on $[-1, 1]$, and normalized so that $P_k(1) = 1$. Assume $k = 2n$ is even and denote by α_{2n} and α_{2n-1} the two smallest positive zeros of P_{2n} and by R_n the polynomial of degree $4n - 8$ defined by

$$R_n(x) = c_n \left(\frac{P_{2n}^{(x)}}{(x^2 - \alpha_{2n}^2)(x^2 - \alpha_{2n-1}^2)}\right)^2, \tag{1.1}$$

where $c_n > 0$ is chosen so that

$$\int_{-1}^{1} R_n(t)\, dt = 1, \qquad n = 2, 3, \ldots. \tag{1.2}$$

The polynomial R_n generates the positive linear polynomial operator

$$L_n(f, x) = \frac{1}{2} \int_{-1}^{1} f(t)\, R_n\left(\frac{t - x}{2}\right) dt, \qquad -1 \leqslant x \leqslant 1. \tag{1.3}$$

This is essentially the operator studied by DeVore in [3, p. 176]. Also note [1, 10]. Let $-1 < x_{1,k} < \cdots < x_{k,k} < 1$ be the zeros of $P_k(x)$ and $\lambda_{\nu,k}$,

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$\nu = 1,\ldots, k$, be the associated Cotes number. In view of the Gauss quadrature formula

$$\int_{-1}^{1} P(t)\,dt = \sum_{\nu=1}^{k} \lambda_{\nu,k} P(x_{\nu,k}), \tag{1.4}$$

valid for all polynomials of degree $\leqslant 2k - 1$, a natural discretization of (1.3) is the positive linear polynomial operator

$$K_n(f, x) = \frac{1}{2} \sum_{\nu=1}^{2n} \lambda_{\nu,2n} f(x_{\nu,2n})\, R_n\left(\frac{x_{\nu,2n} - x}{2}\right). \tag{1.5}$$

This method of discretizing (1.3) is similar to an approach taken by Bojanic and Shisha [2] for discretizing positive trigonometric convolution operators. See also [4, 7, 8]. The purpose of this note is to consider saturation and an asymptotic formula for (1.5).

2. Degree of Approximation

THEOREM 1. *Let* $e_i(x) = x^i$, $i = 0, 1, 2$, *and, for* $0 < \delta < 1$, *let* $I_\delta = [-\delta, \delta]$. *The* $\{K_n\}$ *is locally saturated on* I_δ *with order* n^{-2}, *trivial class* $T(K_n) = \{l : l \text{ is linear on } I_\delta\}$ *and saturation class* $S(K_n) = \{f : f' \in \mathrm{Lip}\, 1 \text{ on } I_\delta\}$.

Proof. The proof is based on the fact that

$$\int_{-1}^{1} t^4 R_n(t)\,dt = O(n^{-4}). \tag{2.1}$$

This is proved in [3, p. 177]. Let $x \in I_\delta$. Using (1.4)

$$\begin{aligned} K_n(e_0, x) &= \frac{1}{2} \sum_{\nu=1}^{2n} \lambda_{\nu,2n} R_n\left(\frac{x_{\nu,2n} - x}{2}\right) \\ &= \frac{1}{2} \int_{-1}^{1} R_n\left(\frac{t - x}{2}\right) dt. \end{aligned}$$

Using (1.2) and (2.1)

$$\begin{aligned} |1 - K_n(e_0, x)| &= \left| \int_{-1}^{1} R_n(t)\,dt - \int_{-(1+x)/2}^{(1-x)/2} R_n(t)\,dt \right| \\ &= \int_{(1-x)/2}^{1} R_n(t)\,dt + \int_{-1}^{-((1+x)/2)} R_n(t)\,dt \\ &\leqslant \int_{(1-\delta)/2}^{1} R_n(t)\,dt + \int_{-1}^{-((1-\delta)/2)} R_n(t)\,dt \\ &\leqslant C_1(\delta)\, M n^{-4}, \end{aligned} \tag{2.2}$$

where $C_1(\delta)$ and M are constants. Next

$$
\begin{aligned}
x - K_n(e_1, x) &= x - \frac{1}{2}\sum_{\nu=1}^{2n} \lambda_{\nu,2n} R_n\left(\frac{x_{\nu,2n} - x}{2}\right) \\
&\quad - x\left(1 - \frac{1}{2}\sum_{\nu=1}^{2n} \lambda_{\nu,2n}(x_{\nu,2n} - x)\, R_n\left(\frac{x_{\nu,2n} - x}{2}\right)\right) \\
&\quad - \frac{1}{2}\sum_{\nu=1}^{2n} \lambda_{\nu,2n}(x_{\nu,2n} - x)\, R_n\left(\frac{x_{\nu,2n} - x}{2}\right) \\
&= x(1 - K_n(e_0, x)) - \frac{1}{2}\int_{-1}^{1} (t - x)\, R_n\left(\frac{t - x}{2}\right) dt \\
&= x(1 - K_n(e_0, x)) - 2\int_{-((1+x)/2)}^{(1-x)/2} tR_n(t)\, dt.
\end{aligned}
$$

Since $tR_n(t)$ is an odd function of t,

$$
\begin{aligned}
\left|\int_{-((1+x)/2)}^{(1-x)/2} tR_n(t)\, dt\right| &\leqslant \int_{(1-|x|)/2}^{(1+|x|)/2} tR_n(t)\, dt \\
&\leqslant \int_{(1-\delta)/2}^{1} tR_n(t)\, dt \\
&\leqslant C_2(\delta)\int_{-1}^{1} t^4 R_n(t)\, dt \\
&\leqslant C_2(\delta)\, Mn^{-4},
\end{aligned}
$$

for some constants $C_2(\delta)$ and M. It follows that

$$
x - K_n(e_1, x) = O(n^{-4}). \tag{2.3}
$$

Using (1.4) and the fact that degree of $(tp-px)^2\, R_n(t - x)$ is $4n - 6$, we obtain

$$
\begin{aligned}
K_n((t - x)^2, x) &= \frac{1}{2}\int_{-1}^{1} (t - x)^2\, R_n\left(\frac{t - x}{2}\right) dt \\
&= 4\left[\int_{-1}^{1} t^2 R_n(t)\, dt - \left(\int_{-1}^{((1+x)/2)} R_n(t)\, dt\right.\right. \\
&\quad \left.\left. + \int_{(1-x)/2}^{1} t^2 R_n(t)\, dt\right)\right].
\end{aligned} \tag{2.4}
$$

Using (1.1) and (1.4),

$$\int_{-1}^{1} t^2 R_n(t)\,dt = \sum_{\nu=1}^{2n} \lambda_{\nu,2n} x_{\nu,2n}^2 R_n(x_{\nu,2n})$$
$$- 2[\lambda_{2n} x_{2n}^2 R_n(x_{2n}) + \lambda_{2n-1} x_{2n-1}^2 R_n(x_{2n-1})], \qquad (2.5)$$

where λ_{2n} and λ_{2n-1} are the Cotes numbers associated with x_{2n} and x_{2n-1}, respectively. There are positive constants C_3, C_4 and a positive integer N such that $n > N$ implies

$$\frac{C_3}{n} \leqslant x_{2n-i} \leqslant \frac{C_4}{n}, \qquad i = 0, 1. \qquad (2.6)$$

This follows from [3, Theorem 1.12]. See also [3, p. 177]. Using (1.2), (2.5) and (2.6), we obtain positive constants C_5 and C_6 such that $n > N$ implies

$$\frac{C_5}{n^2} \leqslant \int_{-1}^{1} t^2 R_n(t)\,dt \leqslant \frac{C_6}{n^2}. \qquad (2.7)$$

Using (2.1), (2.4) and (2.7) we obtain positive constants $C_7(\delta)$, $C_8(\delta)$ and a positive integer $N(\delta)$ such that for $n \geqslant N(\delta)$ and $x \in I_\delta$,

$$\frac{C_7(\delta)}{n^2} \leqslant K_n((t-x)^2, x) \leqslant \frac{C_8(\delta)}{n^2}. \qquad (2.8)$$

Finally

$$K_n((t-x)^4, x) = \frac{1}{2}\int_{-1}^{1} (t-x)^4\, R_n\left(\frac{t-x}{2}\right) dt$$

and it follows from (2.1) that

$$K_n((t-x)^4, x) = O(n^{-4}). \qquad (2.9)$$

Theorem 1 now follows from (2.2), (2.3), (2.8), (2.9), and Theorem 5.3, Lemma 5.2 and Theorem 5.5 of [3].

THEOREM 2. *If f is bounded on $[-1, 1]$ and f'' exists at the fixed point $x \in (-1, 1)$, then*

$$\lim_{n\to\infty} \frac{K_n(R_1, x)}{T_n^{[2]}(x)} = \frac{f''(x)}{2}.$$

where

$$R_1(t) = f(t) - f(x) - f'(t)(t-x)$$

for $-1 < t < 1$ *and* $T_n^{[2]}(x) = K_n((t-x)^2, x)$.

Proof. Let $r > 1$ and $T_n^{[2]}(x) = K_n((t-x)^4, x)$. Since f is bounded on $[-1, 1]$, there exists a positive number $T = T(r, f)$ such that

$$K_n(|f|^r, x) \leqslant T K_n(e_0, x) \leqslant T, \qquad n = 2, 3, \ldots . \tag{2.10}$$

Choose $1 < r' < 2$ such that $r^{-1} + (r')^{-1} = 1$. Using (2.8) and (2.9), we obtain a positive constant, L, such that

$$0 \leqslant \frac{(T_n^{[4]}(x))^{1/r'}}{T_n^{[2]}(x)} \leqslant \frac{Ln^2}{n^{4/r'}} . \tag{2.11}$$

Theorem 2 follows from (2.10), (2.11) and Theorem 2 of [5].

Remarks. The following considerations show that K_n can be used to approximate on an arbitrary interval $I = [a, b]$.

Let $f \in C[a, b]$ and let g be the linear transformation which maps I onto $I_\delta = [-\delta, \delta]$. Let $y \in I$ and $g(y) \in I_\delta$. According to the theorem of Shisha and Mond [6],

$$\begin{aligned} |K_n(f \circ g^{-1}(t), g(y)) - f(y)| &= |K_n(f \circ g^{-1}(t), x) - f \circ g^{-1}(x)| \\ &\leqslant (1 + \|K_n(e_0)\|_{I_\delta})\, w(f \circ g^{-1}, \beta_n) \\ &\quad + \|f \circ g^{-1}\|_{I_\delta} \cdot \|K_n(e_0) - 1\|_{I_\delta}, \end{aligned}$$

where $w(f \circ g^{-1}, \cdot)$ is the modulus of continuity of $f \circ g^{-1}$ on $[-1, 1]$ and

$$\beta_n = \|K_n((t-x)^2, .)\|_{I_\delta}^{1/2} = O(n^{-1}).$$

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